TOTAL LICT DOMINATION IN GRAPHS

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Abstract: For any graph G, the lict graph $n\ G\ = J$ of a graph G is the graph whose vertex set is the union of the set of edges and set of cut vertices of G in which two vertices are adjacent if and only if corresponding members are adjacent or incident. A set D is a total dominating set , if $N\ D\ = V$ or equivalently, if for every vertex $v \in V$, there exists a vertex $u \in S, u \neq v$ such that u is adjacent to v.The total domination number $\gamma_t\ G$ equals the minimum cardinality of total dominating set of G.A dominating set D' of J is a total dominating set if $N\ J\ = V\left[n\ G\ \right]$ and the minimum cardinality of D' is total domination number of $n\ G$ and is denoted by $\gamma_{tn}\ G$ .In this paper, many bounds on $\gamma_{tn}\ G$ were obtained in terms of vertices, edges and other different parameters of G but not in terms of elements of J. Further we develop its relation with other different domination parameters.

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1. Introduction

In this paper, all the graphs considered here are simple, finite, non-trivial, undirected and connected. As usual p and q denote, the number of vertices and edges of a graph G. In this paper, for any undefined terms or notations can be found in Harary

[3].

As usual, the maximum/minimum degree of a vertex in G is denoted by $\Delta(G)/\delta(G)$. The degree of an edge $e = uv$ of G is defined as $\deg e = \deg u + \deg v - 2$ and $\delta'(G)(\Delta'(G))$ is the minimum(maximum) degree among the edges of G.

For any real number x, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the greatest integer not greater than x.

A vertex(edge) cover in a graph G is a set of vertices that covers all the edges (vertices) of G. The vertex(edge) covering number $\alpha_0(G)$ ($\alpha_1(G)$) is a minimum cardinality of a vertex (edge) cover in G. The vertex (edge) independence number $\beta_0(G)(\beta_1(G))$ is the maximum cardinality of independent set of vertices

(edges) in G.

A vertex of degree one is called an end vertex and its neighbor is called support vertex. A vertex v of G is called a cutvertex if removing it from G increases the number of components of G.

A subdivision of edge $e = uv$ of a graph G is the replacement of the edge e by a path uvw. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$.

A line graph $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent (that is , are incident with a comman vertex).

We begin by recalling some standard definition from domination theory.

A set D of a graph $G=(V,E)$ is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The study of domination in graphs was begun by Ore [7] and Berge [1].

A set F of edges in a graph $G=(V,E)$ is called an edge dominating set of G if every edge in $E-F$ is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set in G. Edge domination number was studied by S.L. Mitchell and Hedetniemi [6].

A set $D\subseteq V[L(G)]$ is said to be dominating set of $L(G)$,if every vertex not in D is adjacent to a vertex in D.The domination number of G is denoted by $\gamma[L(G)]$ is the minimum cardinality of dominating set .

A set D is a total dominating set,if $N(D)=V$ or equivalently, if for every vertex $v\in V$,there exists a vertex $u\in S, u\neq v$ such that u is adjacent to v.The total domination number $\gamma_t(G)$ equals the minimum cardinality of total dominating set of G.This concept was introduced by Cockayne,Dawes and Hedetniemi [2].The concept of domination in graphs with its many variations is now well studied in graph theory.(see [4],[5]).

Analogously, we define total domination number in lict graph as follows.

A dominating set D' of $n\ G = J$ is said to be total dominating set if $N\ D' = V\left[n\ G\right]$ or equivalently, if for every vertex $v \in V\left[n\ G\right]$,there exists a vertex $u \in D', u \neq v$ such that u is adjacent to v in $n\ G$ .The total domination number of $n\ G$ is denoted by $\gamma_{tn}\ G$ and is the minimum cardinality of a total dominating set in $n\ G$.

In this paper, many bounds on $\gamma_{tn}\ G$ were obtained in terms of vertices,edges of G .Also we establish total domination number of a lict graph $n\ G$ and express the results with other different domination parameters of G .

2. RESULTS

Initially we begin with Total domination number of lict graph of some standard graphs, which are straight forward in the following theorem.

Theorem 1:

(i) For any cycle C_p with $p \geq 3$ vertices,

$$\gamma_{tn}\ C_p = \frac{p}{2} \qquad \text{if } p \equiv 0 \mod 4 \quad .$$

$$= \frac{p}{2} + 1 \quad \text{if } p = 6 + 4n, n = 0,1,2,3......, i$$

$$= \left\lceil \frac{p}{2} \right\rceil \qquad \text{if } p \text{ is odd .}$$

(ii) For any bipartite graph K_{p_1, p_2} with $p_1 \leq p_2$ vertices,

$$\gamma_{tn}\left[K_{p_1, p_2}\right] = p_1$$

(iii) For any star $K_{1,p}$ with $p > 2$ vertices,

$$\gamma_{tn}\ K_{1,p}\ = 2.$$

Theorem 2. A total lict dominating set $D' \subseteq V[n\ G\ ]$ is minimal if for each vertex $v \in D'$, one of the following condition holds

a) There exists a vertex $u \in V[n\ G\ ] - D'$ such that $N\ u\ \cap D' = v$.

b) v is not an isolated vertex in $\langle D' \rangle$.

c) $\langle\ V[n\ G\ ] - D'\ \cup\ v\ \rangle$ is connected.

Proof: Suppose D' is a minimal total lict dominating set of G and there exists a vertex $v \in D'$ such that v does not hold only of the above conditions. Then for some vertex w ,the set $D_1 = D' \cup\ w$ forms a total lict dominating set in G by condition a and b .Also by c , $\langle V[n\ G\ ] - D' \rangle$ is disconnected. This implies that D_1 is total lict dominating set of G, a contradiction.

Conversely suppose $\forall\ v \in D'$,one of the above statements hold. Further if D is not minimal, then there exists a vertex $v \in D'$ such that $D' -\ v$ such that u dominates v .That is $u \in N\ v$ .Hence v does not satisfy a and b , hence it must satisfy c .Then there exists a vertex $u \in V[n\ G\ ] - D'$ such that $N\ u\ \cap D' =\ v$.

Since $D' -\ v$ is total lict dominating set in G, then there exists a vertex $x \in N\ u\ \cap D'$ where $x \neq v$, a contradiction to the fact $N\ u\ \cap D' =\ v$.

Clearly D' is a minimal total lict dominating set in G.

Theorem 3 : For any connected p, q graph G, $\gamma_{tn}\ G\ \leq \gamma_t\ G\ + \delta\ G$.

Proof: Let $D = \{v_1, v_2, v_3, ..., v_n\}$ be the dominating set of G and $V' = V(G) - D$ be the set such that $H \subseteq V'$ with the minimum set of vertices. Suppose $\langle D \rangle$ has no isolates ,then D itself is a dominating set of G .consider some $v_i \in H$ such that $\forall\ v_j \in D, v_i, v_j \in E(G)$ and $\langle D \cup \{v\} \rangle$ has no isolates, then $D \cup \{v_i\}$ is a total a dominating set . Hence $|D \cup \{v_i\}| = \gamma_t(G)$ .Further consider $E = \{e_1, e_2, e_3, ..., e_n\}$; $C = \{c_1, c_2, c_3, ..., c_n\}$ be the set of edges and cut vertices in G .In lict graph $n(G)$, $V[n(G)] = [E(G) \cup C(G)]$. By representing each element of E as $H = \{u_1, u_2, u_3, ..., u_n\}$ and $J = \{u'_1, u'_2, u'_3, ..., u'_n\}$ of C. Clearly $V[n(G)] = H \cup J$ .Let $H' \subset H$ and $J' \subset J$ be the set of vertices of $n(G)$ such that $\forall\ u_i$ and $u'_i \in \{H' \cup J'\}$ are adjacent to atleast one vertex of $V[n(G)] - \{H' \cup J'\}$ and $\langle H' \cup J' \rangle$ has no isolates ,then $H' \cup J'$ is a total dominating set of $n(G)$ .Suppose a vertex $x \in H'$ or $x \in J'$ such that $\langle \{H' \cup J'\} - x \rangle$ has an isolates. Then $\{H' \cup J'\}$ is a minimal total dominating set of $n(G)$ .Hence $|H' \cup J'| = \gamma_{tn}(G)$ .Since $D \cup \{v_i\}$ is a $\gamma_t(G)$ set , Suppose there exists a vertex v with minimum degree $\delta(G)$ .In $n(G)$ the set is incident to v gives $\delta(e) \in V[n(G)]$ such that $|H' \cup J'| \le |D \cup \{v_i\}| + \delta(v)$ .Thus $\gamma_{tn}(G) \le \gamma_t(G) + \delta(G)$.

Corollary 1: For any graph G if ,

(i) $G \cong W_p$, then $\gamma_{tn}(W_p) \le \left\lfloor \frac{p}{2} \right\rfloor$.

(ii) $G \cong K_{1,p}$, $p \ge 2$ then $\gamma_{tn}(K_{1,p}) = 2$.

Proof: For the condition (i) : If $G \cong W_p$, and $u_1 \in \Delta(W_p)$ .Then $\deg(u_1) = p - 1$.Let

$D' \in V[n(G)]$ and is a total dominating set $n(G)$ such that

$$D' = \{v_1, v_2, v_3, \ldots\ldots\ldots, v_k\} \text{ if } p \text{ is even}$$

$$= \{v_1, v_2, v_3, \ldots\ldots\ldots, v_{k-1}\} \text{ if } p \text{ is odd}$$

Be the total dominating set of $n(W_p)$ .Since the incident edges of u_1 forms a complete induced subgraph in $n(G)$ and any two vertices of D' dominates atleast four vertices in $n(W_p)$,hence it follows that $|D'| \geq \left\lfloor \frac{p}{2} \right\rfloor$.Thus $\gamma_{tn}(W_p) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

For the condition (ii) : If $G \cong K_{1,p}$, $p \geq 2$.Then in this case $n(K_{1,p}) \cong K_{1+p}$. Clearly $\gamma_{tn}(K_{1,p}) = 2 = \gamma_t(K_{p+1})$.

The next Theorem gives the lower bound for $\gamma_{tn}(G)$.

Theorem 4 : For any connected (p,q) graph G , $\gamma_{tn}(G) \leq \left\lfloor \frac{2p}{\Delta(G)} \right\rfloor$

Proof: Let $E = \{e_1, e_2, e_3, \ldots\ldots\ldots\ldots, e_n\}$ and $C = \{c_1, c_2, c_3, \ldots\ldots\ldots\ldots, c_n\}$ be the edge set and cutvertex set of G respectively. In $n(G)$,$V[n(G)] = E(G) \cup C(G)$ .We consider the following cases.

Case i : Suppose G is a tree with $p \geq 3$ vertices ,then in $n(G)$ each block is complete and every cutvertex of $n(G)$ lies on exactly two blocks. Let $C = \{v_1, v_2, v_3, \ldots\ldots\ldots\ldots, v_n\}$ be the set cutvertices in $n(G)$ .Now we consider $C_1 = \{v_1, v_2, v_3, \ldots\ldots\ldots\ldots, v_i\}$; $1 \leq i \leq n$ and $C_2 = \{v_1, v_2, v_3, \ldots\ldots\ldots\ldots, v_j\}$; $1 \leq j \leq n$ such that

$C_1, C_2 \subseteq C$ .Further $\forall\ v_k \in N\ \ C_2$ where $\forall\ v_k \in C_i$, let $H = V\left[n\ \ G\ \right] -\ C_1 \cup C_2$ and $\langle H \rangle$ has no isolates .Thus $\left|\ C_1 \cup C_2\ \right| \leq \left\lfloor \frac{2p}{\Delta} \right\rfloor$,which gives $\gamma_{tn}\ \ G\ \leq \left\lfloor \frac{2p}{\Delta\ \ G} \right\rfloor$.

Case ii : Suppose G is not a tree, then exists atleast one cycle in G .Let e cycle edge in G with maximum edge degree. Now a set $D' =\ v_1, v_2, v_3, ..., v_n$ such that $V\left[n\ \ G\ \right] - D' = D_1, \forall\ v_1 \in D_1$ is adjacent to atleast one vertex of D' .Thus D' is a minimal dominating set. Suppose $\langle D' \rangle$ has no isolates then $\left|D'\right| = \gamma_{tn}\ \ G\ \leq \left\lfloor \frac{2p}{\Delta} \right\rfloor$.

From the above two cases we have $\gamma_{tn}\ \ G\ \leq \left\lfloor \frac{2p}{\Delta\ \ G} \right\rfloor$.

The next Theorem relates $\gamma_{tn}\ \ G$ in terms of vertices and maximum degree of G .

Theorem 5: For any connected p, q graph G with $p \geq 3$ vertices,

$$\gamma_{tn}\ \ G\ \leq P - \Delta\ \ G\ + 1.$$

Proof: Let v be a vertex of maximum degree in G .Let $V =\ v_1, v_2, v_3, ..., v_n$ be the vertex set of G and some $v_i \in C\ \ G\ \ ; 1 \leq i \leq n$,where $C\ \ G$ is the cutvertex set .

Further let D be the dominating set of $n\ \ G$ .Suppose $V_1 = V\left[n\ \ G\ \right] - D$ and $D_1 \in N\ \ D$ where $V_1 \in N\ \ D$ and $D_1 \subseteq V_1$ .Then $D \cup D_1$ forms a total dominating set in $n\ G$ which implies $\left|D \cup D_1\right| \leq V\ \ G\ - \Delta\ \ G\ + 1$.

Clearly $\gamma_{tn}\ \ G\ \leq P - \Delta\ \ G\ + 1$.

The next Theorem relates $\gamma_{tn}\ \ G$ and $\gamma_t\left[L\ \ G\ \right]$.

Theorem 6 : For any non-trivial connected p,q graph $G, \gamma_{tn}\ G \geq \gamma_t\left[L\ G\right]$

Proof: Let D'' be the dominating set of $L\ G$ and $V_1 = V\left[L\ G\right] - D''$ such that $V_1 \in N\ D''$ .Suppose $D_1 \subseteq V_1$ and $D_1 \in N\ D''$; then $D'' \cup D_1$ forms a minimal total dominating set set of $L\ G$ .Further let D be the dominating set of $n\ G$ and let $D_2 \subseteq V_1$ and $D_2 \in N\ D$ then $D \cup D_2$ forms a minimal total dominating set of $n\ G$ .Since $L\ G \subseteq n\ G$, then $\forall\ e_i \in E\ G = V\left[L\ G\right]$; $1 \leq i \leq n, \forall\ e_i \cup c_i\ \in$ $\left[E\ G \cup C\ G\right] = V\left[n\ G\right]$, which gives $e_i \subseteq e_i \cup c_i, \forall\ e_i, c_i \in G$. Clearly $D'' \cup D_1\ \subseteq\ D \cup D_2$ .Thus $|D'' \cup D_1| \leq |D \cup D_2|$.Hence $\gamma_t\left[L\ G\right] \leq \gamma_{tn}\ G$.

In the next Theorem, we obtain the an upper bound for total domination number $n\left[S\ G\right]$.

Theorem 7: For any connected p,q graph $G, \gamma_{tn}\left[S\ G\right] \leq 2\ p - \beta_1$,where β_1 is the edge independence number G.

Proof: Suppose $B = e_1, e_2, e_3, \ldots\ldots., e_n$ be the maximum independent set of edges in G such that $|B| = \beta_1$.Then B is an edge dominating set of G.Let w_i be the vertex set of $S\ G$ which are incident to the edges of B.Further let V' be the set of vertices of G which are incident with any edge of B.If $V' = \phi$,then the corresponding to the edges of B, the vertex set $D' = u_1, u_2, u_3, \ldots\ldots\ldots\ldots., u_n$ forms a total dominating set of $n\left[S\ G\right]$,such that $|D'| \leq 2B$.Hence $\gamma_{tn}\left[S\ G\right] \leq 2\ p - \beta_1$ .Otherwise since B is an edge dominating set of $G, \langle B \rangle$ is independent. Then $D' = u_1, u_2, u_3, \ldots\ldots\ldots\ldots., u_k$;

$1 \leq k \leq n$ forms a minimal total dominating set of $n\left[S\ G\right]$ such that

$\left|D'\right| \leq 2\beta_1 + 2k = 2\beta_1 + 2\ \ p - 2\beta_1\ \ .$

Clearly $\gamma_{tn}\left[S\ \ G\ \right] \leq 2\ \ p - \beta_1\ \ .$

Next we obtain the following characterization.

Theorem 8: For any connected $\ p, q\ $ graph G, if

(i) $G \cong K_p$ then $\gamma_{tn}\left[S\ \ K_p\ \right] = 2\left\lceil \frac{p}{2} \right\rceil$.

(ii) $G \cong K_{p_1,p_2}$, $p_1 \leq p_2$ then $\gamma_{tn}\left[S\ \ K_{p_1,p_2}\ \right] = 2p_2$.

Proof: For (i) suppose $G \cong K_p$, then in this case $\left|D'\right| = \left\lceil \frac{p}{2} \right\rceil$,where D' is total dominating set of $n\ \ G$ by corollary [1] , it follows that $\gamma_{tn}\left[S\ \ K_p\ \right] \leq 2p - 2\left\lceil \frac{p}{2} \right\rceil = 2\left\lceil \frac{p}{2} \right\rceil$.

For (ii) Let $\ X_1, X_2\ $ be the partition of K_{p_1,p_2} with $\left|X_1\right| = p_1$ and $\left|X_2\right| = p_2$.By theorem 7, we have $\gamma_{tn}\left[S\ \ G\ \right] \leq 2\ \ p - \beta_1\ \ = 2\ \ p_1 + p_2\ \ - 2p_1 = 2p_2$.Further any total dominating set $n\left[S\ \ K_{p_1,p_2}\ \right]$ must contain atleast $2p_1$ vertices of X_2 and $2\ \ p_2 - p_1\ $ vertices for dominating the vertices incident with remaining $\ p_2 - p_1$: $\left|D'\right| \geq 2p_1 + 2\ \ p_2 - p_1\ \ = 2p_2$ and hence $\gamma_{tn}\left[S\ \ K_{p_1,p_2}\ \right] = 2p$.

The following theorem relates $\gamma_{tn}\ \ G\ $ and $\gamma'\ \ G\ $.

Theorem 9: For any connected $\ p, q\ $ graph G with $p > 2$ vertices, $\gamma_{tn}\ \ G\ \geq \gamma'\ \ G\ $.

Proof: Let $E = \ e_1, e_2, e_3, ..., e_n\ $ be the edge set of G and $C = \ c_1, c_2, c_3, ..., c_n\ $ be the set of cutvertex in G such that $V[n(G)] = E(G) \cup C(G)$. Let $F = \{e_1, e_2, e_3, \ldots, e_i\}$; $\forall e_i$, where $1 \le i \le n$ be the minimal edge dominating set of G such that $|F| = \gamma'(G)$. Since $E(G) \subseteq V[n(G)]$, then every edge $e_i \in F$; forms a dominating set D' in $n(G)$. Suppose $D'' = E(G) - F \subseteq V[n(G)]$ and $V_1 = V[n(G)] - D'$ where $V_1 \in N(D')$ and $D'' \in N(D')$ such that $D' \cup D''$ forms a minimal total dominating set in $n(G)$. Clearly $|F| \subseteq |D' \cup D''|$. Thus $\gamma'(G) \le \gamma_{tn}(G)$.

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